MATH 4010 | 12/2/2015

Tutorial 5

Eg 1. Let C_0 be the space of all sequences converging to zero, its norm is given by $\|\cdot\|: x=\{\xi_k\} \longmapsto \sup_{k\geq 1} |\xi_k|$.

Prove that $(C_0)^* = \ell^1$

Pf: (i) $\forall\, y=\{\eta_j\}\in \ell^1$, if $x=\{\xi_j\}\in C_0$, then

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \leq \sup_j |\xi_j| \sum_{j=1}^{\infty} |\eta_j| \leq \|x\|_{C_0} \|y\|_{\ell^1} < +\infty$$

Hence, $\forall\, x=\{\xi_k\}\in C_0$, we can define

$$f(x) = \sum_{j=1}^{\infty} \xi_j \eta_j \ ,$$

It is clear that f is linear, and

$$|f(x)| \leq \|y\|_{\ell^1} \|x\|_{C_0}, \ \forall x\in C_0$$

So, $\|f\| \leq \|y\|_{\ell^1} < +\infty$.

On the other hand, we construct $x^{(n)} = \{\xi_j^{(n)}\}$ as

$$\xi_j^{(n)} = \begin{cases} |\eta_j|/\eta_j, & \text{if } j\leq n \text{ and } \eta_j \neq 0 \\ 0, & \text{if } j>n \text{ or } \eta_j = 0, \end{cases} \quad x^{(n)} \in C_0$$

Then $f(x^{(n)}) = \sum_{j=1}^{\infty} \xi_j^{(n)} \eta_j = \sum_{j=1}^{n} |\eta_j|$

Thus $\sum_{j=1}^{n} |\eta_j| = |f(x^{(n)})| \leq \|f\| \|x^{(n)}\|_{C_0} = \|f\|$

Let $n\to\infty$, we have $\|y\|_{\ell^1} = \sum_{j=1}^{\infty} |\eta_j| \leq \|f\|$

Therefore, for any $y\in \ell^1$, there exist a $f \triangleq f_y \in (C_0)^*$ s.t. $\|f\| = \|y\|_{\ell^1}$

Define $T: \ell^1 \to (C_0)^*$ by $T(y) = f_y$, $\forall\, y\in \ell^1$,

where $f_y(x) = \sum_{j=1}^{\infty} \xi_j \eta_j$ with $x=\{\xi_j\}$, $y=\{\eta_j\}$

It is easy to check T is linear. Since $\|T(y)\| = \|f_y\| = \|y\|_{\ell^1}$

T is isometric.

(ii) Now we show that T is surjective. Let $g\in (C_0)^*$.

Since $e_j = (0,\cdots,0,\underset{\substack{\uparrow \\ j\text{-th}}}{1},0,\cdots) \in C_0$, we can set $g(e_j) = \eta_j$

Define $x^{(n)} = \{\xi_j^{(n)}\}$ as above, we have

$$g(x^{(n)}) = g(\xi_j^{(n)} e_j) = \sum_{j=1}^{n} \xi_j^{(n)} g(e_j) = \sum_{j=1}^{n} |\eta_j|$$

Then $\sum_{j=1}^{n} |\eta_j| = |g(x^{(n)})| \leq \|g\| \|x^{(n)}\|_{c_0} = \|g\| < +\infty$.

Let $n \to \infty$, $\sum_{j=1}^{\infty} |\eta_j| < +\infty$. So $y \in l'$.

It sufficies to show that $g = T(y) = f_y$. Indeed,

$\forall x = \{\xi_k\} \in c_0$, $x = \sum_{j=1}^{\infty} \xi_j e_j$. By the continous of g,

$g(x) = \sum_{j=1}^{\infty} \xi_j g(e_j) = \sum_{j=1}^{\infty} \xi_j \eta_j = f_y$.

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Eg 2. Let C be the space of all convergent sequenes with norm $\|\cdot\| = \sup_{k\geq 1} |\xi_k|$.

Prove that $(C)^* = l'$

Pf: (i) $\forall y = \{\eta_j\} \in l'$. If $x = \{\xi_j\} \in C$, then $\xi_j \to a$ for some $a \in \mathbb{R}$ as $j \to \infty$

Since $|a| \leq \|\{\xi_j\}\|_c$, one has

$$|\eta_1 a| + \sum_{j=1}^{\infty} |\eta_{j+1} \xi_j| \leq \|x\|_c \{ |\eta_1| + \sum_{j=1}^{\infty} |\eta_{j+1}| \} = \|x\|_c \|y\|_{l'} < +\infty.$$

Hence, $\forall x \in C$, we can define

$$f(x) = \eta_1 a + \sum_{j=1}^{\infty} \eta_{j+1} \xi_j$$

It is clear that f is linear and $|f(x)| \leq \|y\|_{l'} \|x\|_c$

So, $f \in (C)^*$ and $\|f\| \leq \|y\|_{l'}$.

On the other hand, we can construct $x^{(n)} = \{\xi_j^{(n)}\}$ as

$$\xi_j^{(n)} = \begin{cases} |\eta_{j+1}| / \eta_{j+1} & \text{if } j \leq n \\ |\eta_1| / \eta_1 & \text{if } j > n \end{cases}$$

Then, $\|x^{(n)}\|_c \leq 1$ and $\lim_{j\to\infty} \xi_j^{(n)} = |\eta_1| / \eta_1$, i.e. $x^{(n)} \in C$

Thus, $$f(x^{(n)}) = \eta_1 \frac{|\eta_1|}{\eta_1} + \sum_{j=1}^{n} \eta_{j+1} \frac{|\eta_{j+1}|}{\eta_{j+1}} + \sum_{j=n+1}^{\infty} \eta_{j+1} \frac{|\eta_1|}{\eta_1}$$

$$= \sum_{j=1}^{n+1} |\eta_j| + \frac{|\eta_1|}{\eta_1} \sum_{j=n+2}^{\infty} |\eta_j|$$

So, $$\sum_{j=1}^{n+1} |\eta_j| \leq \left| f(x^{(n)}) - \frac{|\eta_1|}{\eta_1} \sum_{j=n+2}^{\infty} |\eta_j| \right|$$

$$\leq |f(x^{(n)})| + \sum_{j=n+2}^{\infty} |\eta_j| \leq \|f\| \|x^{(n)}\|_c + \sum_{j=n+2}^{\infty} |\eta_j|$$

Let $n \to \infty$, one has $\sum_{j=1}^{\infty} |\eta_j| \leq \|f\|$

Therefore, we can define

$T: l' \to (C)^*$ s.t. $T(y) = f_y$ with

$$f_y(x) = \eta_1 \lim_{j\to\infty} \xi_j + \sum_{j=1}^{\infty} \eta_{j+1} \xi_j, \quad \forall x \in C$$

It is clear that T is linear and $\|T(y)\| = \|f_y\| = \|y\|_{\ell^1}$

So, T is isometric.

(ii) Now, we show that T is surjective. Let $g \in (C)^*$.

Set $\eta_j = g(e_{j-1})$ $j = 2, 3, \cdots$

Define $x^{(n)} = \{\xi_j^{(n)}\}$ as

$$\xi_j^{(n)} = \begin{cases} |\eta_{j+1}|/\eta_{j+1} & \text{if } j \leq n \\ 0 & \text{if } j > n \end{cases}$$

Then $x^{(n)} \in C$ and $\|x^{(n)}\| \leq 1$

$$g(x^{(n)}) = \cancel{\sum_{j=1}^{n+1}} g\left(\sum_{j=2}^{n+1} \xi_j^{(n)} e_{j-1}\right) = \sum_{j=2}^{n+1} \xi_j^{(n)} g(e_{j-1}) = \sum_{j=2}^{n+1} |\eta_j|$$

Thus, $\sum_{j=2}^{n+1} |\eta_j| \doteq |g(x^{(n)})| \leq \|g\| \, \|x^{(n)}\|_C \leq \|g\|$

Let $n \to \infty$, $\sum_{j=1}^{\infty} |\eta_j| \leq \|g\| < +\infty$.

So, one has $y \in \ell^1$ no matter how to choose η_1.

It sufficies to show that $T(y) = g = f_y$. Indeed,

$\forall\, x = \{\xi_j\} \in C$ with $\lim_{j\to\infty} \xi_j = a$.

Define $x^{(n)} = \{\xi_j^{(n)}\}$ as $\xi_j^{(n)} = \begin{cases} \xi_j & \text{if } j \leq n \\ a & \text{if } j > n \end{cases}$

Then $x^{(n)} \in C$ and $\|x - x^{(n)}\|_C = \sup_{j > n} |\xi_j - a| \to 0$ as $n \to \infty$

Thus $x^{(n)} \to x$ in C, so that $g(x^{(n)}) \to g(x)$.

Therefore $g(x) = \lim_{n\to\infty} g(x^{(n)}) = \lim_{n\to\infty} \left\{ a g(e) + \sum_{j=1}^{n} (\xi_j - a) g(e_j) \right\}$

$$= a g(e) + \sum_{j=1}^{\infty} (\xi_j - a)\, \eta_{j+1}$$

$$= a \eta_1 + \sum_{j=1}^{\infty} \xi_j \eta_{j+1}$$

where $\eta_1 = g(e) - \sum_{j=1}^{\infty} \eta_{j+1}$, $e = (1, 1, \cdots, 1, \cdots)$

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Eg 3. $(c_0)^* = l^1$, where c_0 is the space of all sequences of scalars converging to zero, with norm $\|\{\xi_k\}\|_{c_0} = \sup_k \|\xi_k\|$.

Pf: (i) $\forall f \in (c_0)^*$, $f(x) = \sum_{k=1}^{\infty} \xi_k f(e_k)$ with $x = \sum_{k=1}^{\infty} \xi_k e_k \in c_0$ with $\xi_k \to 0$ $_{k\to+\infty}$.

Set $\eta_k = f(e_k)$, then $f(x) = \sum_{k=1}^{\infty} \xi_k \eta_k$.

It suffices to show that $\{\eta_k\} \in l^1$ and $\|\{\eta_k\}\|_{l_1} \leq \|f\|$

Indeed, we can construct $x^{(n)} = \{\xi_k^{(n)}\}$ as

$$\xi_k^{(n)} = \begin{cases} \frac{|\eta_k|}{\eta_k} & \text{if } k \leq n \text{ and } \eta_k \neq 0 \\ 0 & \text{if } k > n \text{ or } \eta_k = 0. \end{cases}$$

Then $f(x^n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \eta_k = \sum_{k=1}^{n} |\eta_k|$, $\because x^{(n)} \in c_0$

and $|f(x^n)| \leq \|f\| \|x^{(n)}\|_{c^0} \leq \|f\|$

So $\sum_{k=1}^{n} |\eta_k| \leq \|f\| < +\infty$, i.e. $\{\eta_k\} \in l^1$

(ii) $\forall y = \{\eta_k\} \in l^1$, define $f: c_0 \to \mathbb{R}$ as

$$f(x) = \sum_{k=1}^{\infty} \xi_k \eta_k, \quad \forall x = \{\xi_k\} \in c_0$$

Then $|f(x)| \leq \sup_k |\xi_k| \sum_{k=1}^{\infty} |\eta_k| \leq \|x\|_{c_0} \|\{\eta_k\}\|_{l^1}$

So $\|f\| \leq \|\{\eta_k\}\|_{l^1} < +\infty$.

Therefore $\|f\| = \|\{\eta_k\}\|_{l^1}$ with $\eta_k = f(e_k)$.